ISRAEL JOURNAL OF MATHEMATICS, Vol. 38, Nos. 1-2, 1981

SINGULAR DIFFERENTIAL OPERATORS AND DISTRIBUTIONS

BY

S. LEIF SVENSSON

ABSTRACT

Differential operators $p(t, \partial) = a_m(t)\partial^m + \cdots + a_0(t)$, where a_m has a zero of finite order at $t = 0$, are studied as operators on the distribution spaces $\mathcal{D}'(R)$ and $\mathcal{E}'(R)$. In particular the kernel of p, operating on $\mathcal{D}'(R)$, is studied in detail by use of asymptotic analysis and a simple formula for its dimension is given. A continuous right inverse for p on $\mathcal{D}'(R)$ is constructed. Necessary and sufficient conditions for this inverse to be two-sided are given. Extensions are made to the spaces $\mathcal{E}(R)$ and $\mathcal{E}'(R)$. Finally some features for operators with more than one singular point are briefly discussed and there is noted a phenomenon — forced propagation of supports — which has important consequences in higher dimensions as a forced propagation of singularities.

0. Introduction

Differential operators

$$p(t, \partial) = a_m(t)\partial^m + \cdots + a_0(t)^{\dagger} \tag{0.1}$$

with smooth coefficients and where $t = 0$ is a zero of finite order for a_m have been studied by Malgrange [5] mainly from the formal point of view. Considered as a formal differential operator working on formal power series expansions at $t = 0$, p is of finite index which is easily computed. The paper of Malgrange also contains some results on the equation $pu = v$ when u and v are germs of distributions at $t = 0$, notably solvability of the equation. The index theorem has also been discussed by Komatsu [4]. Hypoellipticity of operators (0.1) has been discussed by Kannai [3]. In the sequel we shall make the *additional assumption* that, unless otherwise stated, the coefficient a_m only vanishes at $t = 0$. This is done in order not to have to make local versions of all statements.

† $\partial = \partial/\partial t$.

Received January 13, 1980 and in revised form June 2, 1980

The present study is aimed at discussing in detail the kernel of p as an operator on $\mathscr{D}' = \mathscr{D}'(R)$. This kernel is, not surprisingly, of finite dimension. Hence, knowing the looks of the kernel, all information regarding p as an operator on distribution spaces is easily obtainable. By using asymptotic expansions (sections 1 and 2) we prove in section 3 a theorem (3.1) which completely characterizes $\ker_{\mathscr{D}'} p$ and then construct (Theorem 3.2) a continuous right inverse for p on $\mathscr{D}'$. This inverse is sometimes a twosided inverse — indeed a remarkable thing for a differential operator. Hence e.g. $p = t^3\partial + 1$ defines a homeomorphism on $\mathscr{D}'$. Corresponding results are valid for the spaces $\mathscr{E}$ and $\mathscr{E}'$ (Theorem 3.3 and Theorem 4.2). When there are more than one singular point one might lose the solvability property (Example 5.2), but also encounter new examples of invertibility. One example is $p = \cos^2 t\partial - 1$, which is invertible on $\mathscr{D}'$.

Operators such as (0.1) offer striking examples of the great differences between using hyperfunction or distribution spaces. The reason for the statements on invertibility, as well as for the theorem on hypoellipticity by Kannai, is that classical (singular) solutions either behave fairly nicely or else are so bad that they are not even distributions. Fortunately enough one can do very well without the latter type of solutions to construct solutions of $pu = v$ in $\mathscr{D}'$.

The most important aspect in considering operators of the type (0.1) is that such operators, though generally depending upon parameters, appear in many cases as localizations of higher order partial differential operators. Hence this paper may be seen as the initial part of a study of degenerating (i.e. not of principal type) partial differential operators.

1. The formal analysis

We introduce the class $\tilde{\mathscr{F}}$ of formal series expansions

$$\tilde{y}(t) = e^{Q(t)}t^{v} \cdot \sum_{\nu=0}^{N} \tilde{b}_\nu(t)(\log|t|)^{\nu}. \tag{1.1}$$

Here Q is a complex polynomial in $t^{-1/q}$ for some positive integer q, vanishing when $t^{-1/q} = 0$. We recall that Q is called the determining factor of $\tilde{y}$. Further v is a complex number (the index of $\tilde{y}$), $\tilde{b}_0, \cdots, \tilde{b}_N$ are all power series expansions in $t^{1/q}$ and at least one $\tilde{b}_\nu$ has a nonvanishing constant term. Subsets of $\tilde{\mathscr{F}}$ are $\tilde{\mathscr{P}}$ — the space of formal Puiseux series expansions — and $\tilde{\mathscr{O}}$ — the space of formal power series expansions.

Without going into formalities or aiming at completeness we note some simple facts concerning $\tilde{\mathscr{F}}$. Two elements in $\tilde{\mathscr{F}}$ are equal if and only if they have the same q, v, N, Q, and $\tilde{b}_0, \cdots, \tilde{b}_N$. Elements in $\tilde{\mathcal{F}}$ may be formally multiplied and, in case of common determining factors and indices differing at most by a rational number, added. In $\tilde{\mathcal{F}}$ we introduce in the obvious way a formal differentiation $\tilde{\partial}$ ($\tilde{\partial} t^a = at^{a-1}$, $\tilde{\partial} \log|t| = t^{-1}$, $\tilde{\partial} e^Q = (\tilde{\partial} Q)e^Q$ etc.). Given elements $\tilde{a}_0, \cdots, \tilde{a}_m$ in $\tilde{\mathcal{P}}$ with $\tilde{a}_m \neq 0$, we introduce the formal differential operator

$$\tilde{p}(t, \tilde{\partial}) = \tilde{a}_m \tilde{\partial}^m + \cdots + \tilde{a}_0.$$

The formal transpose of $\tilde{p}$ is defined by

$${}^t\tilde{p}(t, \tilde{\partial})\tilde{y} = (-1)^m \tilde{\partial}^m (\tilde{a}_m \tilde{y}) + \cdots + \tilde{a}_0 \tilde{y}.$$

We write

$$\tilde{p} t^\alpha = t^{\alpha - \chi(\tilde{p})} \cdot \mathrm{ind}_{\tilde{p}}(\alpha) + \tilde{c},$$

where, for the index $v(\tilde{c})$ of $\tilde{c}$, $v(\tilde{c}) > \alpha - \chi(\tilde{p})$. Here $\mathrm{ind}_{\tilde{p}}$ is the indicial polynomial of $\tilde{p}$, a polynomial of degree $\rho(\tilde{p})$. Some more or less obvious properties of $\tilde{p}$ are collected in a lemma.

Lemma 1.1. *With $\tilde{p}$ as above one has*

(a) $\chi(\tilde{p}) = \max_j (j - v(\tilde{a}_j))$,
(b) $\rho(\tilde{p}) = \max\{j : j - v(\tilde{a}_j) = \chi(\tilde{p})\}$,
(c) $\chi({}^t\tilde{p}) = \chi(\tilde{p})$, $\rho({}^t\tilde{p}) = \rho(\tilde{p})$,
(d) $\mathrm{ind}_{{}^t\tilde{p}}(\alpha) = \mathrm{ind}_{\tilde{p}}(\chi(\tilde{p}) - 1 - \alpha)$.

Proof. If $\tilde{a}_j = a_{j0} t^{v(\tilde{a}_j)} + \mathrm{hot}$ (higher order terms), one has

$$\tilde{a}_j \tilde{\partial}^j t^\alpha = a_{j0} \alpha(\alpha - 1) \cdots (\alpha - j + 1) t^{v(\tilde{a}_j) + \alpha - j} + \mathrm{hot}.$$

Hence (a) follows immediately. Let $J = \{j : j - v(\tilde{a}_j) = \chi(\tilde{p})\}$. Then

$$\tilde{p} t^\alpha = \left(\sum_{j \in J} a_{j0} \alpha(\alpha - 1) \cdots (\alpha - j + 1) \right) t^{\alpha - \chi(\tilde{p})} + \mathrm{hot},$$

and (b) follows. Further, one has $(-1)^j \tilde{\partial}^j (\tilde{a}_j t^\alpha) = (-1)^j a_{j0} \tilde{\partial}^j t^{v(\tilde{a}_j) + \alpha} + \mathrm{hot}$ and hence

$$\begin{aligned} {}^t\tilde{p} t^\alpha &= \sum_{j \in J} a_{j0} (-1)^j \tilde{\partial}^j t^{v(\tilde{a}_j) + \alpha} + \mathrm{hot} \\ &= \sum_{j \in J} a_{j0} (-1)^j \tilde{\partial}^j t^{j + \alpha - \chi(\tilde{p})} + \mathrm{hot} \\ &= t^{\alpha - \chi(\tilde{p})} \cdot \mathrm{ind}_{\tilde{p}}(\chi(\tilde{p}) - 1 - \alpha) + \mathrm{hot}. \end{aligned}$$

Hence (c) and (d) follow. Thereby the proof is concluded.

We introduce a formal integration $\tilde{\int}$ on $\tilde{\mathscr{F}}$ by putting

$$\tilde{\int} t^{v}(\log|t|)^{k}=\begin{cases} t^{v+1}(\log|t|)^{k}-k\tilde{\int} t^{v}(\log|t|)^{k-1}/(v+1), & v\neq -1,\\ (\log|t|)^{k+1}/(k+1) & \text{if } v=-1.\end{cases}$$

Extending $\tilde{\int}$ by linearity it only remains to define $\tilde{\int}\tilde{z}$ when $\tilde{z}=e^{Q}\tilde{y}$, where Q is a nonvanishing determining factor and where $\tilde{y}$ has a vanishing determining factor. In that case we put

$$\tilde{\int} e^{Q(t)}\tilde{y}(t)=\sum_{j=0}^{\infty} e^{Q(t)}(-1)^{j}(\tilde{\partial}Q)^{-1}\tilde{r}^{j}\tilde{y}(t),$$

where $\tilde{r}\tilde{y}=\tilde{\partial}((\tilde{\partial}Q)^{-1}\tilde{y})$. Since Q is a nonconstant polynomial in $t^{-1/q}$ for some positive integer q, the operator $\tilde{r}$ increases index by at least $1/q$ and hence the infinite sum defines an element in $\tilde{\mathscr{F}}$. It should be observed that while $\tilde{\partial}\tilde{\int}$ equals the identity operator on $\tilde{\mathscr{F}}$, $\tilde{y}-\tilde{\int}\tilde{\partial}\tilde{y}$ equals the coefficient of t^{0} in the expansion of $\tilde{y}$.

It should be noted that given a formal differential operator $\tilde{p}$, there always exists a formal fundamental set $\tilde{y}_1,\cdots,\tilde{y}_m$ of elements in $\tilde{\mathscr{F}}$ for $\tilde{p}$. Thus $\tilde{p}\tilde{y}_j=0$, $j=1,\cdots,m$, and the formal Wronskian $\mathrm{Det}(\tilde{\partial}^{j}\tilde{y}_k)$ is a nonvanishing element in $\tilde{\mathscr{F}}$. For the general proof we refer to Cope [2] and for the construction to Sirovich [6]. It is noteworthy that in the definition (1.1) applied to $\tilde{y}_j$ we may take $q=0$ if Q vanishes. Another fact of importance is that the number of solutions in a formal fundamental set with vanishing determining factor equals the degree $\rho(\tilde{p})$ of the indicial equation.

We shall have use for the index theorem by Malgrange [5], Komatsu [3] concerning formal operators $\tilde{p}$ with coefficients from $\tilde{\mathscr{O}}$: Considered as an operator on $\tilde{\mathscr{O}}$, $\tilde{p}$ is of finite index equal to $\chi(\tilde{p})$.

2. Asymptotic expansions

After the formal analysis of section 1, this section is devoted to the investigation and establishment of asymptotic expansions of certain classes of functions. Also some classes of distributions, defined by aid of expansions, will be discussed.

In the sequel, the functions t^{α} and $\log|t|$ will be considered and so will truncations of expansions in $\tilde{\mathscr{F}}$. We shall not make any notational distinction between those functions and their formal counterparts, since the meaning will always be obvious. For the function t^{α} we choose once and for all some fixed branch e.g. the principal branch.

Assume that $y \in C^\infty(R\setminus\{0\})$ and let $\tilde{y} \in \tilde{\mathcal{F}}$. We write, in the prevailing way of notation,

$$y \sim \tilde{y} \qquad \text{when } t \to 0 \tag{2.1}$$

to denote that there is a sequence $\{c_n\}_1^\infty$ such that $c_n \to \infty$ when $n \to \infty$ and such that for every n

$$e^{-Q(t)}(y(t) - \tilde{y}_n(t)) = O(tc_n) \qquad \text{when } t \to 0. \tag{2.2}$$

Here Q is the determining factor of $\tilde{y}$ and $\tilde{y}_n$ is the truncation of $\tilde{y}$ after the first n terms. When σ is either of $+1$ or -1 we write $y \sim \tilde{y}$ when $t \to 0$, $\operatorname{sgn}(t) = \sigma$ to denote that (2.2) is valid when $\operatorname{sgn}(t) = \sigma$. The class $\mathcal{F}$ ($\mathcal{F}_\sigma$) is defined to consist of all functions y which are infinitely smooth for $t \neq 0$ ($\operatorname{sgn}(t) = \sigma$) and which are such that for some $\tilde{y} \in \tilde{\mathcal{F}}$, $\partial^j y \sim \tilde{\partial}^j \tilde{y}$ when $t \to 0$ ($\operatorname{sgn}(t) = \sigma$) for all $j = 0, 1, 2, \cdots$. The expansions $\tilde{y}$ are not uniquely determined by the corresponding y : s, unless we require the expansions to be, if possible, nonvanishing. This will be taken as a convention for the remainder of the paper.

Let y, z belong to $\mathcal{F}$ ($\mathcal{F}_\sigma$) and let $\tilde{y}, \tilde{z}$ be corresponding expansions in $\tilde{\mathcal{F}}$. Then it is clear that ∂y and yz belong to $\mathcal{F}$ ($\mathcal{F}_\sigma$) with expansions $\tilde{\partial}\tilde{y}$ and $\tilde{y}\tilde{z}$ respectively. If, moreover, $\tilde{y}$ and $\tilde{z}$ have the same determining factor and indices differing only by a rational number, it follows that $y + z$ is in $\mathcal{F}$ with corresponding expansion $\tilde{y} + \tilde{z}$. A simple but very useful observation is that a function y in $\mathcal{F}$ is in $C^\infty(R)$ if and only if it admits a power series expansion $\tilde{y}$.

The properties of integrals of elements in $\mathcal{F}$ are discussed in the following two lemmas.

LEMMA 2.1. *Let y be integrable on* $\operatorname{sgn}(t) = \sigma$, *let Q be a determining factor, and assume that* $y(t) = O(t^\lambda)$ *when* $t \to 0$, $\operatorname{sgn}(t) = \sigma$ *for some* $\lambda > -1$. *Then*

$$e^{-Q(t)} \int_a^t e^{Q(s)} y(s)ds = O(t^{\lambda+1}) \qquad \textit{when } t \to 0, \quad \operatorname{sgn}(t) = \sigma, \tag{2.3}$$

where $\operatorname{sgn}(a) = \sigma$ *in case* $\operatorname{Re} Q(t) \to \infty$ *when* $t \to 0$, $\operatorname{sgn}(t) = \sigma$, *and* $a = 0$ *otherwise.*

PROOF. We assume that $\sigma = 1$. Consider first the case when $\operatorname{Re} Q(t) \to \infty$ when $t \to 0$, $t > 0$. Since Q is a polynomial in $t^{-1/q}$ for some positive integer q, it follows that $\operatorname{Re} Q$ is strictly decreasing when t is small and positive and even that $\exp(-\operatorname{Re} Q(t))$ and $\exp(\operatorname{Re} Q(2t) - \operatorname{Re} Q(t))$ both tend to zero faster than any power of t when $t \to 0$, $t > 0$. We split the integral of (2.3) into three parts and estimate for δ sufficiently small each part in the following way:

$$\left| \int_a^\delta \exp(Q(s) - Q(t)) y(s) ds \right| \leqq C_1 \exp(-\operatorname{Re} Q(t)),$$

$$\left| \int_\delta^{2t} \exp(Q(s) - Q(t)) y(s) ds \right| \leqq C_2 \exp(\operatorname{Re} Q(2t) - \operatorname{Re} Q(t)),$$

$$\left| \int_{2t}^{t} \exp(Q(s) - Q(t)) y(s) ds \right| \leqq C_3 \int_t^{2t} | y(s) | ds.$$

Hence each part is $O(t^{\lambda+1})$ when $t \to 0$, $t > 0$. In case $\operatorname{Re} Q(t)$ remains bounded from above as $t \to 0$, (2.3) follows immediately with $a = 0$. This remark concludes the proof.

Lemma 2.2. *Let $y \in \mathcal{F}_\sigma$ and let $\tilde{y}$ be a corresponding, if possible nonvanishing, expansion in $\tilde{\mathcal{F}}$. Let a be a real number with* $\operatorname{sgn}(a) = \sigma$. *Then there is a number $c_{a,y}$ such that the function*

$$t \to \int_a^t y(s) ds - c_{a,y}$$

is in $\mathcal{F}_\sigma$ and such that

$$\int_a^t y(s) ds - c_{a,y} \sim \tilde{\int} \tilde{y} \qquad \text{when } t \to 0, \quad \operatorname{sgn}(t) = \sigma. \tag{2.4}$$

In case the determining factor Q of $\tilde{y}$ is such that $\operatorname{Re} Q(t) \to \infty$ *when* $t \to 0$, $\operatorname{sgn}(t) = \sigma$, *we may choose* $c_{a,y} = 0$. *Otherwise $c_{a,y}$ is uniquely determined by a and y.*

Proof. We assume that $\sigma = 1$. First it should be observed that in case the determining factor Q of y vanishes, the expansion (2.4) is a consequence of the construction of the finite part integral of Hadamard:

$$c_{a,y} = \text{f.p.} \int_a^0 y(s) ds.$$

We omit the details of proof in this well known case. In case Q does not vanish, we observe that it is no restriction to assume that the interval of integration in (2.4) is so small that ∂Q never vanishes in it. Then the operator ${}^t r = -(\partial Q)^{-1} \partial$ is defined on the interval of integration, ${}^t r e^Q = -e^Q$, and, furthermore, if we write $y = e^Q z$, where $z(t) = O(t^\lambda)$ when $t \to 0$, $t > 0$, it follows that

$$rz(t) = O(t^{\lambda + 1/q}) \qquad \text{when } t \to 0, \quad t > 0, \tag{2.5}$$

where q is some positive integer and r the transpose of ${}^t r$. (Cf. the discussion following the definition of $\tilde{\int}$.) Further, it is no restriction to assume that y vanishes near a. In case $\operatorname{Re} Q(t) \to \infty$ when $t \to 0$, $t > 0$, integration by parts yields (observe that y vanishes near a), if we again write $y = e^Q z$,

$$\int_a^t e^{Q(s)} z(s)\,ds = \sum_{j=0}^{n} e^{Q(t)}(-1)^j (\partial Q(t))^{-1} r^j z(t) + (-1)^{n+1} \int_a^t e^{Q(s)} r^{n+1} z(s)\,ds. \tag{2.6}$$

Comparing (2.6) to the result of Lemma 2.1 and the definition of $\tilde{\int}$ we see that (2.4) follows with $c_{a,y} = 0$. In case $\operatorname{Re} Q$ is bounded from above for small positive t, integration by parts still yields (2.6). However, for n sufficiently large, $e^Q r^{n+1} z$ will be integrable on the interval $[0, a]$ and

$$\int_a^t e^{Q(s)} r^{n+1} z(s)\,ds = \int_a^0 e^{Q(s)} r^{n+1} z(s)\,ds + \int_0^t e^{Q(s)} r^{n+1} z(s)\,ds.$$

In the last integral we may continue integrating by parts as in (2.6), now with $a = 0$, and hence the result follows. The uniqueness result for $c_{a,y}$ is immediate.

In view of Lemma 2.2 it makes sense to define, following the classical finite part terminology by Hadamard,

$$c_{a,y} = \text{f.p.} \int_a^0 y(s)\,ds. \tag{2.7}$$

Here we assume, of course, that $c_{a,y}$ is chosen to be zero if $\operatorname{Re} Q(t) \to \infty$ when $t \to 0$, $\operatorname{sgn}(t) = \operatorname{sgn}(a)$, and we assume y to be nonvanishing if possible. In (2.7) a may take infinite values as well as finite. We also define

$$\text{f.p.} \int_{-\infty}^{\infty} y(s)\,ds = \text{f.p.} \int_0^{\infty} y(s)\,ds + \text{f.p.} \int_{-\infty}^{0} y(s)\,ds$$

and extend the definition to cases when the singular point is not necessarily $t = 0$ and where there are more than one singular point, in ways that are obvious.

If $y \in \mathcal{F}_{+1}$, the finite part integral

$$\text{f.p.} \int_0^{\infty} y(s)\phi(s)\,ds,$$

where ϕ is a test function, either vanishes for any choice of ϕ or else defines a distribution (cf. Lemma 2.2) which uniquely defines y for $t > 0$ and hence may be identified with Hy, where H is the Heaviside function. In the same way $(1-H)y$ and y may, under proper circumstances, be defined as distributions.

Before discussing derivatives of distributions Hy, defined above, it is convenient to introduce still another class of distributions, or, rather, new notations for an old class of distributions. If $y \in \mathscr{F}$ we define the distribution $y\delta_0$ by

$$(2.8)\qquad (y\delta_0)(\phi) = \begin{cases} 0 \text{ if } \tilde{y} \text{ has nonvanishing determining factor,} \\ \text{the coefficient of } t^0 \text{ in } \tilde{y}\tilde{\phi} \text{ otherwise.} \end{cases}$$

Here δ_0 denotes the Dirac measure at the origin and the reason for the notation is, of course, that if $y \in C^\infty$ then $y\delta_0$ retains its meaning as the product of a smooth function and a distribution. Clearly, a necessary condition in order that $y\delta_0$ should not vanish is that $\tilde{y}$ has a vanishing determining factor and nonpositive rational index. The distribution $y\delta_0$ has its support at $t=0$ and, conversely, any distribution with support at $t=0$ may be represented as a distribution (2.8) since a simple computation yields

$$(t^{-j})\delta_0 = ((-1)^j/j!)\delta_0^{(j)}, \quad j \geqq 0.$$

If $\psi \in C^\infty$ then

$$(2.10)\qquad \psi(y\delta_0) = (\psi y)\delta_0.$$

A simple computation yields

$$(2.11)\qquad \partial(y\delta_0) = ((\partial - t^{-1})y)\delta_0,$$

and hence

$$(2.12)\qquad \partial((ty)\delta_0) = (t\partial y)\delta_0.$$

From (2.10) and (2.12) it follows that for any smooth differential operator p

$$(2.13)\qquad p((ty)\delta_0) = (tpy)\delta_0.$$

We observe, recalling the remark following the definition of $\tilde{\int}$, that

$$(2.14)\qquad (y\delta_0)(\phi) = \tilde{y}\tilde{\phi} - \tilde{\phi} - \tilde{\int}\tilde{\partial}(\tilde{y}\tilde{\phi}).$$

When differentiating distributions defined by finite part integrals it is convenient to introduce an operator S working on differential operators and being defined by

$$(2.15)\qquad S(p)y = p(\log|t|\cdot y) - \log|t|\cdot py.$$

Hence $S(p)$ is the commutator of p and (multiplication by) $\log|t|$ i.e. a differential operator of order one less than p. It follows directly that

$$S(a)=0 \quad \text{if } a \text{ is a function}, \qquad S(\partial)=t^{-1}, \tag{2.16}$$

$$S(rq)=S(r)q+rS(q) \quad \text{if } r, q \text{ are differential operators.} \tag{2.17}$$

Defining

$$S^0(p)=p, \quad S^{n+1}(p)=S(S^n(p)) \qquad \text{if } n \geqq 0 \tag{2.18}$$

one gets, by induction over n,

$$p(y(\log|t|)^n)=\sum_{\nu=0}^{n}\binom{n}{\nu}(\log|t|)^{n-\nu}S^\nu(p)y, \qquad n\geqq 0 \tag{2.19}$$

and

$$tS^{n+1}(\partial^{j+1})=(n+1)S^n(\partial^j)+t\partial S^{n+1}(\partial^j), \qquad n\geqq 0, \quad j\geqq 0. \tag{2.20}$$

Now we are prepared for discussing derivatives of finite part integrals.

Lemma 2.3. *Let $y \in \mathscr{F}$. Then, in the sense of distributions*

$$p(Hy(\log|t|)^n)=Hp(y(\log|t|)^n)+(n+1)^{-1}(tS^{n+1}(p)y)\delta_0 \tag{2.21}$$

for any $n\geqq 0$ and any smooth linear differential operator p.

Proof. Obviously it suffices to prove (2.21) when $p=\partial^j$, so we proceed by induction after j. For $j=0$ the whole matter is trivial. To prove (2.21) when $p=\partial$ we note that for $z\in\mathscr{F}$

$$\int_\infty^t z\partial\phi dt - \text{f.p.}\int_\infty^0 z\partial\phi dt \sim \tilde{\int} \tilde{z}\tilde{\partial}\tilde{\phi} \tag{2.22}$$

or, integrating by parts,

$$z(t)\phi(t)-\int_\infty^t(\partial z)\phi dt-\text{f.p.}\int_\infty^0 z\partial\phi dt\sim\tilde{\int}\tilde{z}\tilde{\partial}\tilde{\phi}. \tag{2.23}$$

However, we have

$$\int_\infty^t(\partial z)\phi dt-\text{f.p.}\int_\infty^0(\partial z)\phi dt\sim\tilde{\int}(\tilde{\partial}\tilde{z})\tilde{\phi}. \tag{2.24}$$

Addition of (2.23) and (2.24) yields

$$z(t)\phi(t) - \text{f.p.}\int_{\infty}^{0} z\partial\phi dt - \text{f.p.}\int_{\infty}^{0}(\partial z)\phi dt \sim \tilde{\int} \tilde{\partial}(\tilde{z}\tilde{\phi}).$$

Here the right member is, by (2.14), $\tilde{z}\tilde{\phi} - z\delta_0(\phi)$ and hence

$$\text{(2.25)} \qquad -\text{f.p.}\int_0^{\infty} z\partial\phi dt = \text{f.p.}\int_0^{\infty}(\partial z)\phi dt + z\delta_0(\phi).$$

Letting $z = y(\log|t|)^n$ we see that the last term of (2.25) vanishes unless $n = 0$ in which case

$$z\delta_0 = y\delta_0 = (tt^{-1}y)\delta_0 = (tS(\partial)y)\delta_0.$$

Hence (2.21) follows for the case $p = \partial$. For the step of induction we have, employing (2.12),

$$\text{(2.26)} \quad \partial^{j+1}(Hy(\log|t|)^n) = \partial(H\partial^j(y(\log|t|)^n) + (n+1)^{-1}(t\partial S^{n+1}(\partial^j)y)\delta_0.$$

Recalling (2.19) and using the result for $j = 1$ we get

$$\partial(H\partial^j(y(\log|t|)^n)$$
$$= H\partial^{j+1}(y(\log|t|)^n) + \left(\sum_{\nu=0}^{n}\binom{n}{\nu} tS^{n-\nu+1}(\partial)S^{\nu}(\partial^j)y(n-\nu+1)^{-1}\right)\delta_0.$$

Taking (2.16) into consideration we now get

$$\partial(H\partial^j(y(\log|t|)^n) = H\partial^{j+1}(y(\log|t|)^n) + (S^n(\partial^j)y)\delta_0.$$

Hence (2.26) reads

$$\partial^{j+1}Hy(\log|t|)^n$$
$$= H\partial^{j+1}(y(\log|t|)^n) + ((n+1)S^n(\partial^j) + t\partial S^{n+1}(\partial^j))(n+1)^{-1}y\delta_0.$$

Employing, finally, (2.20) we get (2.21) for $p = \partial^{j+1}$ and hence the lemma is proved.

Given a linear differential operator p of type (0.1) we may form a corresponding formal differential operator $\tilde{p}$ just by performing the Taylor series expansion of the coefficients. We recall the classical method, introduced by G. D. Birkhoff [1], of proving, for a given formal fundamental set $\tilde{y}_1, \cdots, \tilde{y}_m$ for p, the existence of a fundamental set $y_1, \cdots, y_m$ for p, such that each $y_i \in \mathcal{F}$ and $y_i \sim \tilde{y}_i$ when $t \to 0$. This method relies upon the solution of singular integral equations by aid of simple estimates as the one given in Lemma 2.1 above. The real case treated here is actually simpler than the complex case which is usually considered, and corresponds simply to the integration along rays in the complex case. Dependence upon parameters may be considered too but then one runs into the same kind of problems of uniformization as in the complex case. We do not go into any details here but accept the fact that for a given formal fundamental set $\tilde{y}_1, \cdots, \tilde{y}_m$ we may always find a corresponding fundamental set for p.

We finish off this section by a simple uniqueness result.

LEMMA 2.4. *Let $y_1, \cdots, y_m$ be a fundamental set in $\mathcal{F}$ for the operator p such that $\tilde{y}_1, \cdots, \tilde{y}_m$ is a formal fundamental set for p. Assume that $\tilde{y}_1, \cdots, \tilde{y}_m$ contains a basis for the set of formal power series solutions of $\tilde{p}\tilde{y} = 0$. Then any C^∞-solution y of $py = 0$ is, for $t > 0$, a linear combination of those y_j : s for which either the determining factor Q_{y_j} is such that* $\operatorname{Re} Q_{y_j}(t) \to -\infty$ *when $t \to 0$, $t > 0$, or else $\tilde{y}_j$ is a power series expansion.*

PROOF. We have for $t > 0$, $y = \Sigma_j c_j y_j$ and hence

$$c_j = \sum_k a_{jk} \partial^k y, \tag{2.27}$$

where (a_{jk}) is the inverse of the Wronskian matrix $(\partial^j y_k)$. Computing determining factors and indices we get $Q_{a_{jk}} = -Q_{y_j}$ and $v(a_{jk}) = v(y_j)$ (modulo a rational number). Hence the right member of (2.27) is in $\mathcal{F}$ with determining factor $Q_y - Q_{y_j}$ and index $v(y) - v(y_j)$ (modulo a rational number). Since y is assumed to be smooth and the left member of (2.27) is a constant it follows that *either* $\operatorname{Re} Q_{y_j}(t) \to -\infty$ when $t \to 0$, $t > 0$, *or* $Q_{y_j} = 0$ and $v(y_j)$ is a rational number, *or* $c_j = 0$. For the case when $Q_{y_j} = 0$ and $v(y_j)$ is rational, finally we observe that solutions of $py = 0$ with vanishing determining factor and rational index form a linear space and that solutions admitting power series expansions form a linear subspace of this space.

3. The operator p on distribution spaces

We shall study the behaviour of a smooth linear differential operator p of type (0.1) with the highest order coefficient a_m having a zero of finite order at $t = 0$, working on the space $\mathcal{D}'(R)$ of distributions on R. In order to do that we consider the transposed equation

$$^t p\phi = \psi, \qquad \psi \in \mathcal{D}(R). \tag{3.1}$$

Clearly, the only candidate for a solution ϕ of (3.1) in $\mathcal{D}(R)^{\dagger}$ is given by

$$\phi(t)=\sum_{j=1}^{m} y_j(t)\int_{\sigma\cdot\infty}^{t} z_j(s)\psi(s)ds \qquad \text{when } \operatorname{sgn}(t)=\sigma. \tag{3.2}$$

In (3.2) $y_1,\cdots,y_m$ is a fundamental set of solutions of $'py=0$ and $z_1,\cdots,z_m$ are the solutions of the system

$$\sum_{j=1}^{m}(\partial^i y_j)z_j=\delta_{i,m-1}a_m^{-1}(-1)^m,^{\dagger\dagger} \qquad i=0,\cdots,m-1. \tag{3.3}$$

Clearly, the function defined by (3.2) has compact support and is infinitely differentiable for $t\neq 0$. Consequently the condition for ϕ to belong to $\mathcal{D}(R)$ is that ϕ admits a formal power series expansion at $t=0$. We shall see that this is equivalent to a finite number of conditions, each expressible in terms of distributions, upon ψ. We choose $y_1,\cdots,y_m$ in such a way that each $y_j\in\mathcal{F}$ and admits an expansion $\tilde{y}_j$, where $\tilde{y}_1,\cdots,\tilde{y}_m$ is a formal fundamental set for $'\tilde{p}$. We may also, without restrictions, assume that a subset of $\{\tilde{y}_1,\cdots,\tilde{y}_m\}$ forms a basis for the space of formal power series solutions of $'\tilde{p}\tilde{y}=0$. Considering the set $z_1,\cdots,z_m$ defined by (3.3) it is immediately seen that

(a) $z_1,\cdots,z_m$ is a fundamental set for p,

(b) each $z_j\in\mathcal{F}$ and $\tilde{z}_1,\cdots,\tilde{z}_m$ is a formal fundamental set for $\tilde{p}$,

(c) for determining factors and indices one has (from solving (3.3) by Cramer's rule)

$$Q_{z_j}=-Q_{y_j}, \qquad v(y_j)+v(z_j) \text{ is a rational number.}$$

We introduce

$$A_\sigma=\{j:\operatorname{Re}Q_{y_j}\to-\infty \text{ when } t\to 0,\ \operatorname{sgn}(t)=\sigma\},$$

$$B_\sigma=\{j:j\notin A_\sigma, \text{ and } \textit{either } Q_{y_j}\neq 0 \textit{ or } Q_{y_j}=0 \text{ and } v(y_j)\notin Q\},$$

$$C=\{j:Q_{y_j}=0 \text{ and } v(y_j) \text{ is rational}\},$$

$$D=\{j:y_j \text{ admits a nonvanishing power series expansion}\}.$$

By Lemma 2.3, $H_\sigma z_j$, where $H_\sigma(t)=H(\sigma t)$ and H is the Heaviside function, belongs to the kernel $\ker_{\mathcal{D}'}p$ of p in $\mathcal{D}'(R)$ when $j\in B_\sigma$. Hence a necessary condition for solvability of (3.1) is

$$\text{f.p.}\int_{\sigma\infty}^{0} z_j(s)\psi(s)ds=0 \qquad \text{when } j\in B_\sigma. \tag{3.4}$$

† $\mathcal{D}(R)=C_0^\infty(R)$.

†† $\delta_{i,j}$ is the Kronecker delta.

From (3.2) and (3.4) we get for $\operatorname{sgn}(t)=\sigma$

$$(3.5)\qquad \begin{aligned}\phi(t)=\sum_{j\in A_\sigma} y_j(t)\int_{\sigma\infty}^{t} z_j(s)\psi(s)ds+\sum_{j\in B_\sigma\cup C} y_j(t)\bigg(\int_{\sigma\infty}^{t} z_j(s)\psi(s)ds\\ -\text{f.p.}\int_{\sigma\infty}^{0} z_j(s)\psi(s)ds\bigg)+\sum_{j\in C} y_j(t)\bigg(\text{f.p.}\int_{\sigma\infty}^{0} z_j(s)\psi(s)ds\bigg).\end{aligned}$$

By Lemma 2.2 and the definition of f.p. (2.7), the first two sums in the right member of (3.5) are in $\mathcal{F}$ and furthermore, by (c) above, they have expansions with vanishing determining factors and rational indices, something which by definition is true also for the last sum of (3.5). Hence it follows that $\phi\in\mathcal{F}$ and

$$(3.6)\qquad \tilde{\phi}=\sum_j \tilde{y}_j\widetilde{\int}\tilde{z}_j\tilde{\psi}+\sum_{j\in C}\tilde{y}_j\bigg(\text{f.p.}\int_{\sigma\infty}^{0} z_j(s)\psi(s)ds\bigg).$$

Here we observe that the first sum of the right member of (3.6) is an element in $\tilde{\mathcal{F}}$ with coefficients that are linear combinations of expressions $u(\psi)$, where the u : s are distributions with support at $t=0$. Hence, comparing (3.6) to a power series expansion we get two more sets of conditions for solvability of (3.1), namely

$$(3.7)\qquad \begin{gathered}\text{f.p.}\int_{\sigma\infty}^{0} z_j(s)\psi(s)ds+u(\psi)=0\\ \text{for some distribution } u \text{ with support at } t=0,\quad j\in C\backslash D\end{gathered}$$

and, denoting by $\mathcal{L}_p$ the space of distributions u with support at $t=0$ and with $pu=0$,

$$(3.8)\qquad u(\psi)=0\quad\text{when } u\in\mathcal{L}_p.$$

Clearly, $\mathcal{L}_p$ may be identified with the kernel of $\tilde{p}$, considered as an operator on $\tilde{\mathcal{O}}'$—the dual space of $\tilde{\mathcal{O}}$. Returning to (3.6) we see that, (3.7) and (3.8) being satisfied, the right member defines *for each* σ a power series expansion. Matching the two expansions we get a final set of conditions

$$(3.9)\qquad \text{f.p.}\int_{-\infty}^{\infty} z_j(s)\psi(s)ds=0\qquad\text{when } j\in D.$$

To sum up we have found the following set of elements in the kernel of p in $\mathcal{D}'(R)$ by aid of conditions for solvability of (3.1)

(i) $H_\sigma z_j$ for $j\in B_\sigma$

(ii) $H_\sigma z_j+u$ for $j\in C\backslash D$ (where $\operatorname{supp} u_j=0$)

(iii) z_j for $j \in D$

(iv) the elements in $\mathscr{L}_p$.

Since the conditions obtained above are both necessary and sufficient for solvability of (3.1) in $\mathscr{D}(R)$, it follows that the elements (i)–(iv) span the kernel of p in $\mathscr{D}'(R)$. Consequently this kernel has a finite dimension which we shall compute. We observe that for each y_j with determining factor zero there are two solutions (of type (i) or (ii)) except when y_j admits a nonvanishing power series expansion in which case there is only one contribution (of type (iii)). The number of $y_j:s$ of this last type equals the dimension of $\ker_{\tilde{\mathcal{O}}}\tilde{p}$. Further there are contributions from the elements in $\mathscr{L}_p$, but by an earlier remark it follows that the dimension of $\mathscr{L}_p$ equals that of $\ker_{\tilde{\mathcal{O}}'}{}^t\tilde{p}$ i.e. the codimension of the image of $\tilde{p}$ in $\tilde{\mathcal{O}}$. Denote by n_1 the number of couples (σ, j), where $\sigma = \pm 1$ and where $\operatorname{Re} Q(t)$ is bounded from above when $\operatorname{sgn}(t) = \sigma$ if Q is the *nonvanishing* determining factor of $\tilde{z}_j$. Clearly this definition is independent of the particular choice of formal fundamental set $\tilde{z}_1, \cdots, \tilde{z}_m$ and n_1 equals the dimension of the space spanned by solutions of type (i) above with nonvanishing determining factors. Since the total number of $y_j:s$ with vanishing determining factors is $\rho(p)$, equal to the degree of the indicial equation for p, we get the dimension

$$n_1 + 2\rho(\tilde{p}) - \dim \ker_{\tilde{\mathcal{O}}}\tilde{p} + \operatorname{codim} \operatorname{im}_{\tilde{\mathcal{O}}}\tilde{p},$$

which by the index theorem (cf. the end of section 1) equals

$$n_1 + 2\rho(\tilde{p}) - \chi(\tilde{p}).$$

We collect the conclusions in a theorem.

THEOREM 3.1. *The operator* p (0.1) *has a finite-dimensional kernel in* $\mathscr{D}'(R)$ *spanned by the distributions* (i)–(iv) *above. We have*

$$\dim \ker_{\mathscr{D}'} p = n_1 + 2\rho(p) - \chi(p), \tag{3.10}$$

where n_1 *is defined above,* ρ *and* χ *in section* 1. *Furthermore the image of* ${}^t p$ *in* $\mathscr{D}(R)$ *is closed and equals the orthogonal space of the kernel of* p *in* $\mathscr{D}'(R)$.

When constructing a basis for the kernel of a given operator, we do not generally have to construct the fundamental set $y_1, \cdots, y_m$ for the transposed operator. Indeed, by using the dimension statement of Theorem 3.1, Lemma 2.3, and the formula (2.13), we get detailed information on the kernel just by inspecting the system used in computing formal solutions of $py = 0$. To compute the solutions u of $pu = 0$ with support at $t = 0$ we employ (2.13). Hence we have to find all $t^{-m}\tilde{y}$, where $\tilde{y} \in \tilde{\mathcal{O}}$, $y_0 \neq 0$, $m > 0$, and such that $\tilde{p}(t^{-m}\tilde{y})$ has index

$v \geqq 0$. Then the solutions are $u = (t^{1-m}\tilde{y})\delta_0$. To compute the solutions of type (ii) above, we take *maximal* sequences

$$\tilde{z}_k = \sum_{j=k}^{n} (\tilde{y}_j/((j-k)!))(\log|t|)^{j-k}, \qquad k = 0, \cdots, n$$

of formal solutions of $\tilde{p}\tilde{z} = 0$. Then, in particular, we get from (2.19)

$$\sum_{\nu=0}^{n-k-\mu} (1/\nu!)S^{\nu}(\tilde{p})\tilde{y}_{\mu+k+\nu} = 0, \qquad \mu = 0, \cdots, n-k. \tag{3.11}$$

Now, a simple computation, by aid of (2.21) shows that if $pz_k = 0$ for $t \neq 0$ and $z_k \sim \tilde{z}_k$ when $t \to 0$, then

$$p(H_\sigma z_k) = \sigma \cdot \left(\sum_{\nu=1}^{n-k+1} (1/\nu!)S^{\nu}(\tilde{p})\tilde{y}_{\nu+k-1} \right) \delta_0. \tag{3.12}$$

If $k > 0$, we get by putting $\mu = 0$ and replacing k by $k-1$ in (3.11)

$$\tilde{p}\tilde{y}_{k-1} + \left(t \cdot \sum_{\nu=1}^{n-k+1} (1/\nu!)S^{\nu}(\tilde{p})\tilde{y}_{\nu+k-1} \right) = 0. \tag{3.13}$$

Combining this with (3.12) we see that

$$H_\sigma z_k + \sigma(t\tilde{y}_{k-1})\delta_0 \tag{3.14}$$

is a solution. Hence there only remains to check the last elements from each sequence, i.e. we should look for an element $\tilde{y}_{-1}$ such that the left member of (3.13) with $k = 0$ has non-negative index. Then (3.14) with $k = 0$ still defines a solution. However, if no such $\tilde{y}_{-1}$ exists, then z_0 only contributes a solution of type (iii). We consider a couple of examples.

EXAMPLE 3.1. Consider the operator

$$p = t^k\partial + a.$$

If $k > 1$ and $a \neq 0$ we have $\rho(p) = \chi(p) = 0$ and hence $\dim \ker_{\mathcal{D}'} p = n_1$. A solution of $py = 0$ is $y = \exp(at^{1-k}/(k-1))$. Hence we get the following cases:

	n_1	Basis for $\ker_{\mathcal{D}'} p$
$\operatorname{Re} a = 0$	2	$H_{+1}y$, $H_{-1}y$
$\operatorname{Re} a > 0$, k odd	0	—
$\operatorname{Re} a < 0$, k odd	2	$H_{+1}y$, $H_{-1}y$
$\operatorname{Re} a \neq 0$, k even	1	$H_\sigma y$, where $\operatorname{Re} a\sigma < 0$.

If $k \geqq 1$ and $a = 0$ we have $\rho(p) = 1$, $\chi(p) = 1 - k$, and $n_1 = 0$. Hence we get $\dim \ker_{\mathscr{D}'} p = 1 + k$. A basis is in this case H_{-1}, H_{+1}, $\delta_0, \cdots, \delta_0^{(k-2)}$. If $k = 1$ and $a \neq 0$ we get $\dim \ker_{\mathscr{D}'} p = 2$. A solution of $py = 0$ is $y = t^{-a}$. Hence a basis is $H_{+1}y$, $H_{-1}y$ unless a is a positive integer, in which case a basis is formed by y, $(ty)\delta_0$.

EXAMPLE 3.2. Consider the confluent hypergeometric operator

$$p = t\partial^2 + (\gamma - t)\partial - \beta.$$

The indicial equation is

$$\mathrm{ind}_p(\alpha) = \alpha(\alpha - (1 - \gamma)) = 0.$$

Since $n_1 = 0$, $\rho(p) = 2$, and $\chi(p) = 1$, we get $\dim \ker_{\mathscr{D}'} p = 3$. We have, if $\tilde{y}_1, \tilde{y}_2$ denotes elements in $\tilde{\mathcal{O}}$, the following types of solutions of $py = 0$:

(a) γ is no integer	$y_1,\ t^{1-\gamma}y_2$
(b) γ, β are integers and $\gamma - 1 \leqq \beta < 0$	(same)
(c) γ, β are integers and $0 \leqq \beta < \gamma - 1$	(same)
(d) not (a)–(c), $\gamma \leqq 1$	$t^{1-\gamma}y_2,\ y_1 + t^{1-\gamma}y_2 \log\lvert t\rvert$
(e) not (a)–(c), $\gamma > 1$	$y_1,\ t^{1-\gamma}y_2 + y_1 \log\lvert t\rvert$

In case (a) we get immediately the distribution solutions y_1, $H_{+1}t^{1-\gamma}y_2$ and $H_{-1}t^{1-\gamma}y_2$, which by the dimension statement also span the kernel. The same is true in case (b) but now because the sum of (3.14) in this case has non-negative index. In case (c) we observe that $(t^{2-\gamma}y_2)\delta_0$ is a nontrivial solution. Hence a basis is y_1, y_2, and $(t^{2-\gamma}y_2)\delta_0$. For the two last cases we have maximal sequences as discussed above with two elements. Observing that y_1 has index zero, we get the basis elements $H_\sigma t^{1-\gamma}y_2$, $\sigma = \pm 1$, and $y_1 + t^{1-\gamma}y_2 \log|t|$ for case (d) and $H_\sigma y_1 + \sigma(t^{2-\gamma}y_2)\delta_0$, $\sigma = \pm 1$, and $y_2 t^{1-\gamma} + y_1 \log|t|$ for case (e).

We shall now use the knowledge of the kernel of p in $\mathscr{D}'(R)$ to construct a continuous right inverse for p.

THEOREM 3.2. *Let Ω_1 be a neighborhood of $t = 0$ and let Ω_2 be an open subset of R such that*

(i) $\Omega_1 \cap \Omega_2 = \varnothing$,

(ii) *if* $\Omega_2 \cap \operatorname{supp} u = \varnothing$† *for some* $u \in \ker_{\mathscr{D}'} p$ *then* $\operatorname{supp} u = \{0\}$.

Then there is a continuous right inverse P_r^{-1} for p on $\mathscr{D}'$ such that for any $\psi \in \mathscr{D}$

(iii) $\operatorname{supp}(\mathrm{id}_{\mathscr{D}}^{\dagger\dagger} - {}^t p\, {}^t p_r^{-1})\psi \subseteq \Omega_1 \cup \Omega_2$,

† $\operatorname{supp} u$ denotes the support of the distribution u.

†† $\mathrm{id}_{\mathscr{D}}$ denotes the identity operator on $\mathscr{D}$.

(iv) $\operatorname{supp}(\mathrm{id}_{\mathscr{D}} - {}^t p\, {}^t p_r^{-1})\psi \subseteq \Omega_2$ *if and only if* $\tilde{\psi} \in \mathrm{im}_{\mathscr{O}}{}^t\tilde{p}$, *where* $\tilde{\psi}$ *is the Taylor expansion of* ψ *at* $t = 0$.

PROOF. Take a basis $u_1, \cdots, u_n$ for $\ker_{\mathscr{D}'} p$ such that $u_{k+1}, \cdots, u_n$ is a basis for the subspace with support at $t = 0$. By (i) and (ii) it follows that we may take functions $\psi_1, \cdots, \psi_n$ in $\mathscr{D}$ such that $\operatorname{supp} \psi_j \subseteq \Omega_1$ for $j = k+1, \cdots, n$ and $\operatorname{supp} \psi_j \subseteq \Omega_2$ for $j = 1, \cdots, k$ and such that

$$u_j(\psi_k) = \delta_{j,k}, \qquad j, k = 1, \cdots, n. \tag{3.15}$$

We put as in (3.2)

$${}^t p_r^{-1}\psi(t) = \sum_j y_j(t) \int_{\sigma\infty}^{t} z_j(s)\Big(\psi(s) - \sum_\nu u_\nu(\psi)\psi_\nu(s)\Big)ds,$$

when $\operatorname{sgn}(t) = \sigma$. Then, by Theorem 3.1, ${}^t p_r^{-1}\psi \in \mathscr{D}$ for any $\psi \in \mathscr{D}$ and ${}^t p_r^{-1}\,{}^t p$ equals the identity operator on $\mathscr{D}$. Furthermore, the mapping ${}^t p_r^{-1}$ is continuous. Indeed this follows either directly by using Lemma 2.1 or by the closed graph theorem which is valid in $\mathscr{D}$. A simple computation yields

$$(\mathrm{id}_{\mathscr{D}} - {}^t p\, {}^t p_r^{-1})\psi = \sum_\nu u_\nu(\psi)\psi_\nu.$$

By (3.15) and the definition of the $\psi_j : s$ (iii) follows at once and we see that

$$\operatorname{supp}\Big(\sum_\nu u_\nu(\psi)\psi_\nu\Big) \subseteq \Omega_2 \tag{3.16}$$

if and only if $u_\nu(\psi) = 0$ for $\nu = k+1, \cdots, n$. Now, $u_{k+1}, \cdots, u_n$ is by definition a basis for the subspace $\mathscr{L}_p$ of $\ker_{\mathscr{D}'} p$, consisting of elements with support at $t = 0$. There is an obvious identification of $\mathscr{L}_p$ and $\ker_{\mathscr{O}'} \tilde{p}$ such that the condition (3.16) is equivalent to the condition that $\tilde{\psi}$ (the Taylor expansion at $t = 0$ of ψ) belongs to the orthogonal space of $\ker_{\mathscr{O}'} \tilde{p}$ i.e. to $\mathrm{im}_{\mathscr{O}}{}^t\tilde{p}$. This proves (iv) and concludes the proof of the theorem.

It should be observed that one may always choose as the Ω_2 of Theorem 3.2 any neighborhood of the infinity in R which is disjoint with Ω_1. A particular consequence is hence that p is surjective on $\mathscr{D}'(R)$. Theorem 3.2 also yields information on p as an operator on $\mathscr{E} = C^\infty(R)$ and on $\mathscr{E}'$.

THEOREM 3.3. *Let the operator* p *be of type* (0.1) *and let* $y_1, \cdots, y_m$ *be a fundamental set for* p *in* $\mathscr{F}$ *such that the corresponding formal fundamental set* $\tilde{y}_1, \cdots, \tilde{y}_m$ *contains a basis for the space of formal power series solutions of* $\tilde{p}\tilde{y} = 0$. *Then*

(i) $\ker_{\mathscr{E}'} p = \ker_{\mathscr{D}'} p \cap \{u \in \mathscr{E}' : \operatorname{supp} u = \{0\}\}$,

(ii) $\ker_{\mathscr{E}} p$ *is spanned by all* $H_\sigma y_j$ *such that* $\operatorname{Re} Q_{y_j}(t) \to -\infty$ *when* $t \to 0$, $\sigma t > 0$, *and by all* $y_j : s$ *such that* $\tilde{y}_j$ *admits a power series expansion,*

(iii) $\operatorname{im}_{\mathscr{E}} p$ *equals the orthogonal space of* $\ker_{\mathscr{E}'} {}^t p$,

(iv) $\operatorname{im}_{\mathscr{E}'} p$ *equals the orthogonal space of* $\ker_{\mathscr{E}} t^p$.

PROOF. (i) is a simple consequence of the fact that $y_1, \cdots, y_m$ are linearly independent functions on any nonempty open subset of R. The statement (ii) follows from Lemma 2.4. To prove (iii) we construct a right inverse as in Theorem 3.2 but with p replaced by ${}^t p$. We write

$$\psi = pp_r^{-1}\psi + (\mathrm{id}_{\mathscr{D}} - pp_r^{-1})\psi = p\psi_1 + \psi_2.$$

Now let $\psi \in \mathscr{D}$ and be orthogonal to $\ker_{\mathscr{E}'} {}^t p$. By (i) and the identification made in the proof of Theorem 3.2 it follows that $\tilde{\psi} \in \operatorname{im}_{\mathscr{O}} \tilde{p}$. Hence, by (iv) of Theorem 3.2, $\operatorname{supp} \psi_2 \subseteq \Omega_2$ for some open set Ω_2 with positive distance to the origin. But then, in solving $p\phi_2 = \psi_2$ one may start integrating at $t = 0$, thus showing that $\psi_2 \in \operatorname{im}_{\mathscr{E}} p$. Hence $\psi \in \operatorname{im}_{\mathscr{E}} p$ and we have proved the nontrivial inclusion of (iii). To prove the nontrivial inclusion of (iv) we take a $v \in \mathscr{E}'$ which also belongs to the orthogonal space of $\ker_{\mathscr{E}} {}^t p$. The Ω_2 of Theorem 3.2 may be chosen in such a way that the complement of Ω_2 is compact, convex, and contains $\Omega_1 \cup \operatorname{supp} v$. Putting $u = p_r^{-1} v$, it follows that $pu = v$ and it only remains to prove that u has compact support. Indeed we prove that the support of u is contained in the complement of Ω_2. Let $\phi \in \mathscr{D}$ with $\operatorname{supp} \phi \subseteq \Omega_2$. Then $u(\phi) = v({}^t p_r^{-1} \phi)$. However, by (iv) of Theorem 3.2, ${}^t p_r^{-1} \phi$ agrees in $\operatorname{supp} v$ with an element in $\ker_{\mathscr{E}} {}^t p$ and hence, by assumption, $v({}^t p_r^{-1} \phi) = 0$ i.e. $u(\phi) = 0$. Hence $\operatorname{supp} u$ is contained in the complement of Ω_2 and therefore compact.

4. Invertibility

In certain cases the right inverse of Theorem 3.2 is really a twosided inverse.

THEOREM 4.1. *The operator* p (0.1) *is invertible on* $\mathscr{D}'$ *if and only if*

(i) $\operatorname{Re} Q(t) \to \infty$ *when* $t \to 0$ *for any possible determining factor of a solution in* $\mathscr{F}$ *of* $py = 0$,

and

(ii) $\chi(p) = 0$.

PROOF. Obviously p is invertible on $\mathscr{D}'$ if and only if $\ker_{\mathscr{D}'} p = 0$, which by (3.10) is equivalent to

(1) (i) above

and

(2) $2\rho(p) - \chi(p) = 0$.

However, the condition (i) includes, in particular, the condition that the indicial equation is of degree zero i.e. $\rho(p) = 0$. Hence the theorem is proved.

There are corresponding statements for the spaces $\mathscr{E}$ and $\mathscr{E}'$.

THEOREM 4.2. *The operator p* (0.1) *is invertible on* $\mathscr{E}$ ($\mathscr{E}'$) *if and only if*

(i) $\operatorname{Re} Q(t)$ *is bounded from below* (*above*) *as* $t \to 0$ *for any determining factor* Q *of a solution in* $\mathscr{F}$ *of* $py = 0$,

(ii) $\chi(p) = 0$,

(iii) $\operatorname{ind}_p(n) \neq 0$ *for any integer* $n \geqq 0$ ($\leqq -1$).

PROOF. Obviously, p is invertible on $\mathscr{E}$ if and only if

(a) $\ker_{\mathscr{E}} p = \{0\}$,

(b) $\operatorname{im}_{\mathscr{E}} p = \mathscr{E}$.

By (iv) of Theorem 3.3 we may replace (b) by

(c) $\ker_{\mathscr{E}'} {}^t p = \{0\}$.

Further, by (i)–(ii) of Theorem 3.3, (a) and (c) may be replaced by

(d) (i) above,

(e) $\alpha = \dim(\ker_{\mathscr{D}'} {}^t p \cap \{u \in \mathscr{D}' : \operatorname{supp} u = \{0\}\}) = 0$,

(f) $\beta = \dim(\ker_{\mathscr{O}} \tilde{p}) = 0$.

However, by the index theorem

$$\beta - \alpha = \chi(p).$$

Moreover (f) is clearly equivalent to (iii). Hence the statement follows for $\mathscr{E}$. Essentially the same argument, or else duality, employing (d) of Lemma 1.1, may be used for the space $\mathscr{E}'$.

It should be noted that an operator which is invertible on $\mathscr{D}'$ is also invertible on $\mathscr{E}$ but never on $\mathscr{E}'$. The invertibility is further discussed in some examples.

EXAMPLE 4.1. The operator (cf. Example 3.1)

$$p = t^k \partial + a$$

has $\chi(p) = 0$ unless $a = 0$ in which case $\chi(p) = 1 - k$. The operator is invertible on $\mathscr{D}'$ if and only if

$$k \text{ is odd}, \quad k > 1, \quad \text{and} \operatorname{Re} a > 0,$$

it is invertible on $\mathscr{E}$ if and only if either

$$k = 1 \text{ and } a \text{ is not an integer} \leqq 0$$

or

$$k \text{ is odd}, \quad k > 1, \quad \text{and } \operatorname{Re} a > 0,$$

or

$$k \text{ is even}, \quad k > 1, \quad \operatorname{Re} a = 0, \quad \text{and } \operatorname{Im} a \neq 0.$$

EXAMPLE 4.2. The *real* operator

$$p = t^6 \partial^2 + \alpha t^3 \partial + \beta$$

is invertible on $\mathscr{D}'$ if and only if $0 < \alpha$, $0 < \beta$. Indeed a simple computation shows that $\chi(p) = 0$ if and only if $\beta \neq 0$ and that the possible determining factors of solutions in $\mathscr{F}$ of $py = 0$ are $Q = -ct^{-2}/2$, where $c^2 + \alpha c + \beta = 0$.

EXAMPLE 4.3. For the operator

$$p = t^2 \partial^2 + at\partial + b,$$

we have $\chi(p) = 0$ and $\operatorname{ind}_p(\alpha) = \alpha(\alpha - 1) + a\alpha + b$. The only possible determining factor is $Q = 0$. In particular

$$p = t^2 \partial^2 + 4t\partial + 2$$

is invertible on $\mathscr{E}$ but not on $\mathscr{E}'$,

$$p = t^2 \partial^2$$

is invertible on $\mathscr{E}'$ but not on $\mathscr{E}$, and

$$p = t^2 \partial^2 + 1/4$$

is invertible on both $\mathscr{E}$ and $\mathscr{E}'$.

For completeness and for comparison we recall the hypoellipticity theorem by Kannai [3]. This theorem could of course easily be proved with the tools that we have developed, but the proof would not differ in any essential ideas from that of Kannai. The theorem says: The operator p (0.1) is hypoelliptic if and only if $|\operatorname{Re} Q(t)| \to \infty$ when $t \to 0$, for any determining factor of a solution in $\mathscr{F}$ of $py = 0$ *and* $a_{\rho(p)} \neq 0$. By hypoellipticity is here meant that any distribution solution of $pu = \phi \in \mathscr{E}$ is itself a function in $\mathscr{E}$. The conditions for hypoellipticity require the classical solutions of $py = 0$ to be either so nice that they are

C^∞-functions or so bad that they are not even distributions. As has been pointed out by Komatsu [4] the situation is quite different in hyperfunction spaces.

5. Generalizations and concluding remarks

An obvious generalization is to include dependence upon parameters. Indeed the whole theory may be repeated for operators (0.1) in R^{n+1} where the coefficients depend upon $(t, x_1, \cdots, x_n)$, and where the formal solutions may be labeled in such a way that they do not change any essential properties nor interfere mutually as $x_1, \cdots, x_n$ vary. We do not make any digression into what the last should mean exactly but only point to the complex case with polar coordinates r, ϕ corresponding to t and x_1. As is well known the Stokes' phenomenon may appear even though the individual formal solutions are well behaved. The higher dimensional problem will be treated for first order operators in a forthcoming paper [7]. A new feature in higher dimensions is that one may lose the property of *local* solvability.

Another case of generalization is that of an operator (0.1) where the highest order coefficient a_m has an isolated but infinite order zero at a point. For that case we also refer to [7].

For the case of more than one singular point we can give a general treatment only to operators of first order. The technique applied in the proof of Theorem 3.1 may then be employed. We do not make any general discussion but only consider a couple of examples.

EXAMPLE 5.1. The operator

$$p = \cos^2 t\partial - 1$$

is invertible on $\mathscr{D}'(R)$. When t lies in an interval $](n-1/2)\pi, (n+1/2)\pi[$ we define for $\psi \in C_0^\infty$

$$\phi(t) = \int_{(n-1/2)\pi}^{t} \exp(\tan\sigma - \tan t)\cos^{-2}\sigma\psi(\sigma)d\sigma.$$

Integrating by parts we get

$$\phi(t) = \psi(t) - \int_{(n-1/2)\pi}^{t} \exp(\tan\sigma - \tan t)\partial\psi(\sigma)d\sigma. \tag{5.1}$$

From the last formula it is easily seen, using Lemma 2.2, that ϕ, properly defined at the zeros of $\cos t$, is a C_0^∞-function. Moreover, if I is the smallest interval $[(k+1/2)\pi, (n+1/2)\pi]$ which contains the support of ψ, then I contains the support of ϕ too. Clearly (5.1) defines a continuous inverse for ${}^t p$ on $\mathscr{D}(R)$ and hence p has a continuous inverse on $\mathscr{D}'$. Moreover, if v is a distribution with compact support and $pu = v$, then u too has compact support, contained in the smallest interval $[(k + 1/2)\pi, n + 1/2)\pi]$ which contains the support of v.

EXAMPLE 5.2. The operator

$$p = t^2(t-1)^2\partial - 2t + 1$$

is not surjective on $\mathscr{D}'(R)$. Indeed a simple computation shows that the determining factor Q for a solution y of ${}^t py = 0$ in $\mathscr{F}$ at $t = 0$ is $Q = -1/t$, while the corresponding determining factor at the point $t = 1$ is $Q_1 = 1/(t-1)$. This means that there is a C_0^∞-solution ϕ of ${}^t p = 0$ and consequently the equation $pu = v$, where $v \in \mathscr{D}'(R)$ has a distribution solution only if v is orthogonal to ϕ. That this condition also is sufficient for solvability is easily seen.

Apart from the phenomenon that singular operators might have continuous inverses on distribution spaces, there is another fact which is important and does not seem to have been recognized before and that is a phenomenon of forced propagation of support from a source. The counterpart in higher dimensions is a forced propagation of singularities.

EXAMPLE 5.3. Consider the operator

$$p = t^2\partial + 1.$$

The kernel of p as an operator on $\mathscr{D}'(R)$ is spanned by $H_{-1}y$, where $y = \exp 1/t$. If $\operatorname{sgn}(a) = +1$ and δ_a is the Dirac measure at a, any solution u of $pu = \delta_a$ in $\mathscr{D}'$ is of the form

$$u(t) = (cH(-t) + H(t-a))\exp 1/t.$$

Hence, contrary to the "normal" cases when one may choose either of the two directions of a characteristic (or both) as direction of propagation from a source, we have in this case no choice but to follow the positive direction from a.

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Lund Universiteits Matematiska Institution
Box 725
S-220 07, Lund, Sweden